Characteristic Polynomial Coefficients and Their Relationship with Eigenvalues, Trace, and Determinants

Given the following matrices, their characteristic polynomials, eigenvalues, and eigenvectors, we request to verify in the characteristic polynomials found (exercise 1 **a)** and **g)**):

1. The relationship between the coefficients of the polynomial and the eigenvalues.
2. The relationship between the coefficients of the polynomial and the trace and determinant of the matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

Characteristic polynomial:

$$p(\lambda) = \lambda^2 - 3\lambda - 4.$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$\lambda_2 = -1, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix}.$$

Characteristic polynomial:

$$p(\lambda) = (1 - \lambda)(\lambda - 3)^2.$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda_2 = \lambda_3 = 3, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solution

First matrix

(1) Relationship between coefficients and eigenvalues:
For a 2×2 matrix, the characteristic polynomial is expressed as

$$p(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

Here,

$$\lambda_1 + \lambda_2 = 4 + (-1) = 3 \quad \text{and} \quad \lambda_1\lambda_2 = 4 \cdot (-1) = -4.$$

This matches the polynomial

$$\lambda^2 - 3\lambda - 4.$$

(2) Relationship with the trace and determinant:
The trace of A is:

$$\text{tr}(A) = 1 + 2 = 3,$$

and the determinant is:

$$\det(A) = 1 \cdot 2 - 2 \cdot 3 = 2 - 6 = -4.$$

We observe that:

$$\text{tr}(A) = \lambda_1 + \lambda_2 \quad \text{and} \quad \det(A) = \lambda_1\lambda_2,$$

confirming the verification.

Second matrix

(1) Relationship between coefficients and eigenvalues:
For a 3×3 matrix, the characteristic polynomial in monomial form is

$$p(\lambda) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + \cdots + (-1)^3\lambda_1\lambda_2\lambda_3.$$

In our case:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 3 + 3 = 7,$$

and

$$\lambda_1\lambda_2\lambda_3 = 1 \cdot 3 \cdot 3 = 9.$$

Expanding $p(\lambda)$:

$$\begin{aligned}(1-\lambda)(\lambda-3)^2 &= (1-\lambda)(\lambda^2 - 6\lambda + 9),\\ &= \lambda^2 - 6\lambda + 9 - \lambda^3 + 6\lambda^2 - 9\lambda,\\ &= -\lambda^3 + 7\lambda^2 - 15\lambda + 9.\end{aligned}$$

Multiplying by -1 to obtain the standard form (with leading coefficient $+1$), we have:

$$\lambda^3 - 7\lambda^2 + 15\lambda - 9.$$

Here, the coefficient of λ^2 is -7, which confirms that $-(\lambda_1 + \lambda_2 + \lambda_3) = -7$, and the constant term is -9, verifying that $(-1)^3\lambda_1\lambda_2\lambda_3 = -9$.

(2) Relationship with the trace and determinant:
The trace of A is:

$$\text{tr}(A) = 1 + 2 + 4 = 7,$$

and the determinant is the product of the eigenvalues:

$$\det(A) = 1 \cdot 3 \cdot 3 = 9.$$

These values match the sum and product of the eigenvalues, respectively, verifying the relationship with the coefficients of the polynomial.